On the Autoconvolution Equation and Total Variation Constraints *

Gunter Fleischer[†] Rudolf Gorenflo[‡] Bernd Hofmann[§]

1991 Mathematics Subject Classifications:

65 J20, 45 G10, 65 R30

Keywords:

autoconvolution, ill-posed problem, discretization,

constrained least-squares approach, bounded total variation

Abstract

This paper is concerned with the numerical analysis of the autoconvolution equation $x * x = y$ restricted to the interval $[0, 1]$. We present a discrete constrained least squares approach and prove its convergence in $L^p(0, 1)$, $1 \leq p < \infty$, where the regularization is based on a prescribed bound for the total variation of admissible solutions. This approach includes the case of non-smooth solutions possessing jumps. Moreover, an adaption to the Sobolev space $H^1(0, 1)$ and some remarks on monotone functions are added. The paper is completed by a numerical case study concerning the determination of non-monotone smooth and non-smooth functions x from the autoconvolution equation with noisy data y.

---

*The research was supported by the Deutsche Forschungsgemeinschaft (DFG) Grant Ho 1454/3-2 and by the Alexander von Humboldt Foundation, Bonn.

†Faculty of Mathematics, Technische Universität Chemnitz–Zwickau, D–09107 Chemnitz, Germany, email: fleische@mathematik.tu-chemnitz.de

‡Department of Mathematics and Computer Science, Free University of Berlin, Arnimallee 2-6, D-14195 Berlin, Germany, email: gorenflo@math.fu-berlin.de

§Faculty of Mathematics, Technische Universität Chemnitz–Zwickau, D–09107 Chemnitz, Germany, email: hofmannb@mathematik.tu-chemnitz.de

1 Introduction

In the paper [8] by GORENFLO and HOFMANN the *nonlinear ill-posed autoconvolution equation*

$$\int_0^s x(s-t)\,x(t)\,dt \;=\; y(s), \quad 0 \le s \le 1, \tag{1.1}$$

on the *finite interval* $[0,1]$ has been analyzed. This autoconvolution problem can be written as an operator equation

$$F(x) \;=\; y \tag{1.2}$$

with the *continuous* nonlinear operator $F : D(F) \subset X \to Y$ defined by

$$[F(x)](s) := [x * x](s) := \int_0^s x(s-t)\,x(t)dt, \quad 0 \le s \le 1, \tag{1.3}$$

and mapping between Banach spaces X and Y with norms $\|.\|_X$ and $\|.\|_Y$, respectively, containing real functions on the interval $[0,1]$. In [8] there have been discussed intrinsic properties of the autoconvolution operator F from (1.3) and conditions for its compactness, injectivity and weak closedness, in particular for the Hilbert space $X = Y = L^2(0,1)$. As a consequence the general theory of Tikhonov regularization became applicable to equation (1.1). The character of ill-posedness in this equation strongly depends on the solution point x and its *local degree of ill-posedness*. Applications of the autoconvolution equation arising in physics and in stochastics are also mentioned in [8].

On the other hand, we discussed in a recent paper (cf. [5]) including numerical results the case that x is considered as a function of the space $L^2(-\infty,\infty)$ possessing a support *supp* $x \subset [0,1]$, where the complete data function $[x*x](s)$ $(0 \le s \le 2)$ is observable. In such a case Fourier transform techniques are applicable and yield some more insight into the behaviour of the autoconvolution equation. However, the knowledge of $[x*x](s)$ for $s > 1$ is not always realistic. Therefore, in the present paper we are going to investigate *stable approximate discretized solutions* to (1.1), where both the function x to be determined and the data function y that can be measured are restricted to arguments from the interval $[0,1]$.

The approximate solution of the autoconvolution equation (1.1) will be based for $Y := L^2(0,1)$ on the restriction of admissible solutions x to *compact subsets* of the domain $D(F)$ with prescribed properties. Provided that F is injective the inverse operator F^{-1} becomes continuous. We will show in Section 2 that a compactification of the autoconvolution equation in $X := L^p(0,1)$ can be based on a prescribed upper bound c for the *total variation* $T(x)$ of solutions x, which are in addition uniformly bounded below and above by positive constants a and b, respectively. This allows us to construct *convergent* discretized solutions also in the case of non-smooth solutions possessing jumps. In this context, we generalize the well-known descriptive regularization approach using the set of monotone functions uniformly bounded below and above as a compact subset in $L^p(0,1)$, $1 \le p < \infty$ (cf. Section 4, [13] and [4]). The total variation bound c plays in our consideration the role of a regularization parameter. In Section 3, the ideas of Section 2 are extended to the Sobolev space case $X := H^1(0,1)$. A brief reference to the case of monotone functions is given in Section 4. The paper is completed by a case study presented in Section 5 that illustrates the theoretical assertions of Section 2. In this case study the behaviour of discretized least-squares solutions to the autoconvolution equation subject to uniform bounds of the total variation is investigated, where both the case of a smooth and of a non-smooth solution are reflected.

2 Discretizing the Autoconvolution Equation under Total Variation Constraints

Let us consider the autoconvolution operator (1.3) between the Banach spaces $X := L^p(0,1)$ for fixed $2 \le p < \infty$ with norm $\|x\|_{L^p(0,1)} = \left(\int_0^1 |x(t)|^p\, dt\right)^{1/p}$ and $Y := L^2(0,1)$. In this context, we define the sets

$$D_\varepsilon^+ := \{x \in L^p(0,1) : x(t) \ge 0 \text{ a.e. in } [0,1],\ \varepsilon = \sup\{\tau : x(t) = 0 \text{ a.e. in } [0,\tau]\}\} \tag{2.1}$$

and

$$R_\varepsilon^+ := \{y \in L^2(0,1) : y(s) \ge 0 \text{ a.e in } [0,1],\ \varepsilon = \sup\{\chi : y(s) = 0 \text{ a.e. in } [0,\chi]\}\}. \tag{2.2}$$

Then we have the following proposition which, because of $L^p(0,1)$ being densely embedded in $L^2(0,1)$, follows from [8, Theorem 1 and Lemma 6] and [5, Proposition 2.5]:

Proposition 2.1 *The autoconvolution operator $F : L^p(0,1) \to L^2(0,1)$ from (1.3) is a continuous nonlinear operator for all $2 \le p < \infty$. In the restricted case $F : D_0^+ \subset L^p(0,1) \to R_0^+ \subset L^2(0,1)$ the operator is injective, but the autoconvolution equation (1.2) is locally ill-posed in the sense of Definition 2.2 in all points $x \in D_0^+$.*

Definition 2.2 *We call the equation (1.2) locally ill-posed in* $x \in D(F)$ *if, for arbitrarily small* $r > 0$ *and balls* $B_r := \{\tilde{x} \in X : \|\tilde{x} - x\|_X \le r\}$*, there is an infinite sequence* $\{x_k\} \subset D(F) \cap B_r(x)$ *with*

$$\|F(x_k) - F(x)\|_Y \to 0, \quad \textit{but} \quad \|x_k - x\|_X \not\to 0 \quad \textit{as} \quad k \to \infty. \tag{2.3}$$

Otherwise the equation is called locally well-posed in $x \in D(F)$*.*

To overcome the difficulties of ill-posedness of a problem under consideration one can restrict the domain $D(F)$ to a subset, which is *compact* in the Banach space X.

For a real function $x(t)$ $(0 \le t \le 1)$ we denote by

$$T(x) := \sup_{0 \le t_0 < t_1 < \ldots < t_{k-1} < t_k \le 1} \sum_{i=1}^{k} |x(t_i) - x(t_{i-1})| \tag{2.4}$$

the *total variation* of the function x on $[0,1]$ and by $T_S(x)$ the analogously defined total variation of x on a closed subinterval $S \subset [0,1]$. Note that the supremum in formula (2.4) is to be taken over all possible finite grids of the form $0 \le t_0 < t_1 < \ldots < t_{k-1} < t_k \le 1$ with an arbitrarily chosen integer k. We consider, for given positive constants a, b and c, where

$$0 < a < b, \tag{2.5}$$

the domain

$$D := \left\{ x : [0,1] \to [a,b], \quad T(x) \le c, \quad \begin{array}{c} \text{x left-continuous for } t \in (0,1], \\ \text{x right-continuous for t=0} \end{array} \right\}. \tag{2.6}$$

For technical reasons we assume that the lower bound a is strictly positive (see the remark after formula (2.21)). Obviously we have $D \subset L^p(0,1)$ for all $1 \le p < \infty$. The requirement of the left- and right-continuity for the functions $x \in D$ is reasonable, since a function of bounded variation has due to [12, Corollary 2, Chap. VIII, § 3] only a countable set of discontinuity points, namely jumps. Therefore, the left limit $\lim_{t \to t_0 - 0} x(t)$ exists in all points of the interval $(0,1]$. In the continuity points t_0 this limit coincides with the value $x(t_0)$. In all other points let be the values of x defined by $x(t_0) := \lim_{t \to t_0 - 0} x(t)$. That means, with respect to $L^p(0,1)$-elements we consider the representative, which is left-continuous in every point $t \in (0,1]$. Moreover let $x(0) := \lim_{t \to 0+0} x(t)$, i.e. we consider no jumps at $t = 0$.

Lemma 2.3 *The domain* D *from (2.5) – (2.6) is a compact subset of* $L^p(0,1)$*,* $1 \le p < \infty$*, and we have* $D \subset D_0^+$*.*

The proof of compactness of D is based on HELLY's theorem (cf. e.g. [12, Chap. VIII, §4]). For the proof ideas we refer to [4, Lemma 4.2]. On the other hand, note that Lemma 2.3 is a corollary of Theorem 2.5 in the paper [1] of ACAR and VOGEL. Namely, the set D from (2.5) – (2.6) is bounded with respect to the BV-norm

$$\|x\|_{BV[0,1]} := \|x\|_{L^1[0,1]} + T(x). \tag{2.7}$$

Based on Lemma 2.3 providing compactness the following well-known Lemma of TIKHONOV will allow us to prove stability results.

Lemma 2.4 *Let $F : D(F) \subset X \to Y$ be a continuous and injective operator between the Banach spaces X and Y with a compact domain $D(F)$. We denote by x^*, for given right-hand side $y^* \in F(D(F))$, the unique solution of the operator equation (1.2). Then for a family of approximate solutions $x_\eta \in D(F)$ the convergence of residual norms*

$$\|F(x_\eta) - F(x^*)\|_Y \to 0 \quad as \quad \eta \to 0 \tag{2.8}$$

implies the convergence of the approximate solutions

$$\|x_\eta - x^*\|_X \to 0 \quad as \quad \eta \to 0. \tag{2.9}$$

A slightly modified version of this theorem and its proof can be found in BAUMEISTER's book [2, p. 18].

In order to obtain numerical approximate solutions, in the sequel we are going to discretize the autoconvolution equation (1.1) – (1.3), where the restriction of F to the compact subset D from (2.5) – (2.6),

$$F : D \subset L^p(0,1) \to L^2(0,1), \tag{2.10}$$

is used. Similar to the discretization methods in [7] and [11], where also a total variation constraint is essential, we subdivide the interval $[0,1]$ into n subintervals I_i of the uniform length $h := 1/n$, where

$$I_i := ((i-1)h, ih] \qquad (i = 1, ..., n).$$

For simplicity we set $T_i(x) := T_{[(i-1)h,ih]}(x)$ for $x \in D$. Moreover, let

$$t_j := \frac{h}{2} + (j-1)h \quad (j = 1, ..., n)$$

denote the midpoints and

$$s_i := ih \quad (i = 1, ..., n)$$

the right endpoints of such intervals.

To discretize the nonlinear integral equation (1.1), for all $i, j = 1, 2, ..., n$ the values $x(t_j)$ and $y(s_i)$ will be approximated by some x_j and y_i, respectively. A *discrete autoconvolution operator*

$$\underline{F} : \mathbb{R}^n \to \mathbb{R}^n \tag{2.11}$$

can be defined by

$$\underline{F}(\underline{x}) := \left(\sum_{j=1}^{i} h x_{i-j+1} x_j \right)_{i=1}^{n}, \quad \underline{x} = (x_1, ..., x_n)^T. \tag{2.12}$$

In its discrete form the autoconvolution equation then reads as

$$\underline{F}(\underline{x}) \ = \ \underline{y}, \quad \underline{y} = (y_1, ..., y_n)^T, \tag{2.13}$$

or as

$$\sum_{j=1}^{i} h x_{i-j+1} x_j = y_i, \quad (i = 1, ..., n). \tag{2.14}$$

The realistic situation that the given data are noisy can be included. Instead of the exact data y_i for the right-hand side we will use perturbed data $\hat{y}_i$, where

$$\|\underline{\hat{y}} - \underline{y}\|_2 \le \delta \tag{2.15}$$

and δ is a fixed upper bound for the noise of the data vector $\underline{\hat{y}} = (\hat{y}_1, ..., \hat{y}_n)^T$. Here we have used the scaled Euclidean norm

$$\|\underline{z}\|_2 := \left(\sum_{i=1}^{n} h z_i^2 \right)^{\frac{1}{2}}$$

for $\underline{z} \in \mathbb{R}^n$. For our further investigations we introduce the restriction operators

$$R : D \subset L^p(0,1) \to \mathbb{R}^n \quad \text{and} \quad Q : F(D) \subset L^2(0,1) \to \mathbb{R}^n$$

by

$$(R(x))_j := x(t_j) \quad (j = 1, ..., n) \tag{2.16}$$

and

$$(Q(y))_i := y(s_i) \quad (i = 1, ..., n), \tag{2.17}$$

as well as the extension operators $E_1 : \mathbb{R}^n \to L^p(0,1)$ and $E_2 : \mathbb{R}^n \to L^2(0,1)$ by

$$(E_1(\underline{x}))(t) := x_j \quad (t \in I_j,\ j = 1, ..., n), \quad (E_1(\underline{x}))(0) := x_1 \tag{2.18}$$

and

$$(E_2(\underline{y}))(s) := y_i \quad (s \in I_i,\ i = 1, ..., n), \quad (E_2(\underline{y}))(0) := y_1. \tag{2.19}$$

We are searching now for an optimal solution vector

$$\underline{x}^{opt} = (x_1^{opt}, ..., x_n^{opt})^T$$

solving the *discrete least-squares problem*

$$\|\underline{F}(\underline{x}) - \underline{\hat{y}}\|_2 \to \min, \quad \text{subject to} \quad \underline{x} \in M, \tag{2.20}$$

where M is defined as

$$M := \left\{ \underline{x} \in \mathbb{R}^n : 0 < a \le x_i \le b \ (i = 1, ..., n),\ \sum_{i=1}^{n-1} |x_{i+1} - x_i| \le c \right\}. \tag{2.21}$$

There exist solutions of (2.20), since M is compact in $\mathbb{R}^n$ and $\|\underline{F}(\underline{x}) - \underline{\hat{y}}\|_2 : \mathbb{R}^n \to \mathbb{R}^1$ is a continuous functional possessing a minimum over M. The condition $0 < a \le x_i \le b$ is more restrictive than the discretized version of $x \in D_0^+$. We require this stronger condition, because we want M to be a compact subset of $\mathbb{R}^n$.

For the vectors $\eta := (\delta, h)^T$, $\underline{x}^{opt} \in M$ and $\underline{\hat{y}}$ we define the piecewise constant function $x_\eta \in D$ by

$$x_\eta(t) := E_1(\underline{x}^{opt})(t) \quad (0 \le t \le 1). \tag{2.22}$$

and the piecewise constant function y_δ by

$$y_\delta(s) := E_2(\underline{\hat{y}})(s) \quad (0 \le s \le 1).$$

Lemma 2.5 *If we define the operator $F_\eta : L^p(0,1) \to L^2(0,1)$ by the formula*

$$[F_\eta(x)](s) := \sum_{j=1}^{i} \int_{I_j} x(s_i - t) x(t) dt \quad (s \in I_i), \tag{2.23}$$

then we have the equation

$$\|F_\eta(\xi) - \zeta\|^2_{L^2(0,1)} = \|\underline{F}(\underline{\xi}) - \underline{\zeta}\|_2^2 \tag{2.24}$$

for all $\xi := E_1(\underline{\xi})$, where $\underline{\xi} := (\xi_1, .., \xi_n)^T \in \mathbb{R}^n$ and all $\zeta := E_2(\underline{\zeta}) \in L^2(0,1)$, where $\underline{\zeta} := (\zeta_1, .., \zeta_n)^T \in \mathbb{R}^n$.

Proof:

$$\begin{aligned}
\|F_\eta(\xi)-\zeta\|^2_{L^2(0,1)} &= \int_0^1 ([F_\eta(\xi)](s)-\zeta(s))^2 ds \\
&= \sum_{i=1}^n \int_{I_i} \left(\sum_{j=1}^i \int_{I_j} \xi(s_i - t)\xi(t)dt - \zeta(s) \right)^2 ds \\
&= \sum_{i=1}^n h \left(\sum_{j=1}^i h\xi_{i-j+1}\xi_j - \zeta_i \right)^2 \\
&= \|\underline{F}(\underline{\xi}) - \underline{\zeta}\|^2_2.
\end{aligned}$$

This proves the lemma ■

Lemma 2.6 *Let $x \in D$ from (2.5) – (2.6). Then we have the estimation*

$$\|F(x) - F_\eta(x)\|_{L^2(0,1)} \le 2hb^2 + 2hbc.$$

Proof: We write

$$\|F(x)-F_\eta(x)\|_{L^2(0,1)} = \left(\sum_{i=1}^n \int_{I_i} \left(\int_0^s x(s-t)x(t)dt - \int_0^{s_i} x(s_i - t)x(t)dt \right)^2 ds \right)^{\frac{1}{2}}. \tag{2.25}$$

Then we can estimate the expression in the inner parentheses by

$$\begin{aligned}
& \left| \int_0^s x(s-t)x(t)dt - \int_0^{s_i} x(s_i-t)x(t)dt \right| \\
\le\ & \left| \int_{s_{i-1}}^s x(s-t)x(t)dt \right| + \left| \int_0^{s_{i-1}} x(s-t)x(t)dt - x(s_i-t)x(t)dt \right| + \left| \int_{s_{i-1}}^{s_i} x(s_i-t)x(t)dt \right| \\
\le\ & hb^2 + \sum_{j=1}^{i-1} \int_{I_j} |x(s-t) - x(s_i-t)||x(t)|dt + hb^2 \\
\le\ & b\sum_{j=1}^{i-1} \int_{I_j} |x(s-t)-x(s_i-t)|dt + 2hb^2. \qquad (2.26)
\end{aligned}$$

Now we substitute $u := s_i - t$, $du := -dt$. For a fixed point $t \in (s_{j-1}, s_j] = I_j$ we obtain $u \in (s_{i-j}, s_{i-j+1}] = I_{i-j+1}$ and in view of $-h \le s - s_i \le 0$

$$s - s_i + u \in (s_{i-j-1}, s_{i-j+1}] = I_{i-j} \cup I_{i-j+1}.$$

Moreover, we can estimate (2.26) by

$$b\sum_{j=1}^{i-1} \int_{I_j} |x(s-t) - x(s_i - t)|dt + 2hb^2$$

$$
\begin{aligned}
&= b\sum_{j=1}^{i-1}\int_{I_{i-j+1}} |x(s-s_i+u)-x(u)|du + 2hb^2 \\
&\le hb\sum_{j=1}^{i-1}(T_{i-j}(x)+T_{i-j+1}(x)) + 2hb^2 \\
&\le hbT(x)+hbT(x)+2hb^2 \\
&\le 2hbc+2hb^2.
\end{aligned}
$$

Finally we substitute this estimation into equation (2.25). This yields the assertion of the lemma ∎

Lemma 2.7 *Under the assumptions stated above we have*

$$
\|F(x_\eta)-F(x^*)\|_{L^2(0,1)} \le 4hb^2+6hbc+2\delta \to 0 \quad as \quad \eta\to 0. \tag{2.27}
$$

Proof (for similar ideas see also [6]): From the triangle inequality we obtain

$$
\|F(x_\eta)-F(x^*)\|_{L^2(0,1)} \le \|F(x_\eta)-F_\eta(x_\eta)\|_{L^2(0,1)}+\|F_\eta(x_\eta)-y_\delta\|_{L^2(0,1)}+\|y_\delta-y\|_{L^2(0,1)}. \tag{2.28}
$$

The right-hand side of (2.28) consists of three terms which we want to estimate one by one: Due to Lemma 2.6 for the first term it holds

$$
\|F(x_\eta)-F_\eta(x_\eta)\|_{L^2(0,1)} \le 2hb^2+2hbc \quad (x_\eta\in D).
$$

To estimate the second term of (2.28) we define $\underline{x}^* := R(x^*)$ as the vector of the function values of the exact solution x^* of the autoconvolution equation (1.1) in the midpoints of the intervals I_i. Since we have $\underline{x}^{opt}$ as the least-squares solution of (2.20), the residual norm of $\underline{x}^*$ cannot be smaller than the residual norm of $\underline{x}^{opt}$. Furthermore, we can apply Lemma 2.5 with $\xi := x_\eta$ and $\zeta := y_\delta$. This yields

$$
\|F_\eta(x_\eta)-y_\delta\|_{L^2(0,1)} = \|\underline{F}(\underline{x}^{opt})-\underline{\hat{y}}\|_2 \le \|\underline{F}(\underline{x}^*)-\underline{\hat{y}}\|_2.
$$

Using the identity

$$
F_\eta(x) = E_2(Q(F(x))) \quad (x\in D),
$$

this allows us to estimate further as follows:

$$
\begin{aligned}
&\|\underline{F}(\underline{x}^*)-\underline{\hat{y}}\|_2 \le \|\underline{F}(\underline{x}^*)-Q(F(x^*))\|_2+\|Q(F(x^*))-\underline{\hat{y}}\|_2 \\
=\ &\|F_\eta(E_1(R(x^*)))-E_2(Q(F(x^*)))\|_{L^2(0,1)}+\|\underline{y}-\underline{\hat{y}}\|_2
\end{aligned}
$$

$$= \left(\sum_{i=1}^{n} \int_{I_i} \left(\sum_{j=1}^{i} \int_{I_j} (\tilde{x}(s_i - t)\tilde{x}(t) - x^*(s_i - t)x^*(t))dt \right)^2 ds \right)^{\frac{1}{2}} + \delta$$

$$\leq \left(\sum_{i=1}^{n} \int_{I_i} \left(\sum_{j=1}^{i} \int_{I_j} |\tilde{x}(s_i - t)||\tilde{x}(t) - x^*(t)| + |x^*(s_i - t) - \tilde{x}(s_i - t)||x^*(t)|dt \right)^2 ds \right)^{\frac{1}{2}} + \delta$$

$$\leq \left(\sum_{i=1}^{n} \int_{I_i} \left(\sum_{j=1}^{i} \int_{I_j} 2bT_j(x^*)dt \right)^2 ds \right)^{\frac{1}{2}} + \delta \ \leq \ 2hbc + \delta,$$

where $\tilde{x} := E_1(R(x^*))$. The last inequalities essentially used Lemma 2.5 with $\xi = E_1(R(x^*)) = \tilde{x}$ and $\zeta = E_2(Q(F(x^*)))$, respectively. Note that we have $\tilde{x}(t) = x^*(t_j)$ for $t \in I_j$ and thereby $|\tilde{x}(t) - x^*(t)| \leq T_j(x^*)$. Taking into account $|y_i - \hat{y}_i| \leq \delta$ and the identity

$$\|E_2(\underline{y})\|_{L^2(0,1)} = \|\underline{y}\|_2,$$

which can easily be proved, we hence can estimate the third term of (2.28) as follows (cf. Lemma 2.6):

$$\begin{aligned} & \|y_\delta - y\|_{L^2(0,1)} \leq \|y - E_2(Q(y))\|_{L^2(0,1)} + \|E_2(Q(y)) - y_\delta\|_{L^2(0,1)} \\ = \ & \|F(x^*) - E_2(Q(F(x^*)))\|_{L^2(0,1)} + \|E_2(Q(y)) - E_2(Q(y_\delta))\|_{L^2(0,1)} \\ = \ & \|F(x^*) - F_\eta(x^*)\|_{L^2(0,1)} + \|Q(y) - Q(y_\delta)\|_2 \leq 2hb^2 + 2hbc + \delta. \end{aligned}$$

Finally we can add the three terms and obtain by (2.28) the inequality (2.27). Evidently, the right-hand side of (2.27) tends to zero as h and δ both tend to zero. This proves the lemma

■

By the result of Lemma 2.7 we can apply Lemma 2.4 to prove in L^p-spaces the convergence of approximate solutions to the autoconvolution equation under total variation constraints.

Theorem 2.8 *Consider the autoconvolution problem (1.1) – (1.3) with $D(F) := D$ from (2.5)-(2.6) and denote by $x^* \in D$, for given right-hand side $y^* \in F(D(F))$, the unique solution of the autoconvolution equation. Then the family of approximate solutions x_η according to (2.22) converges to the solution x^* of (1.2):*

$$\|x_\eta - x^*\|_{L^p(0,1)} \to 0 \quad as \quad \eta \to 0 \quad for\ all \quad 1 \leq p < \infty. \tag{2.29}$$

Proof: In the case $p \geq 2$ based on Lemma 2.7 the Lemma 2.4 immediately yields the convergence property (2.29), since the autoconvolution operator $F : D \subset L^p(0,1) \to L^2(0,1)$

is continuous and injective. Furthermore, D is a compact subset in $L^p(0,1)$ because of Lemma 2.3. For $1 \le p < 2$ the norm $\|\cdot\|_{L^p(0,1)}$ is 'weaker' than the norm $\|\cdot\|_{L^2(0,1)}$. This ensures the convergence condition (2.29) also in this case ■

By using the method of TIKHONOV regularization in Hilbert spaces X and Y the minimizers x_α of the auxiliary extremal problems

$$\|F(x) - y\|_Y^2 + \alpha \|x\|_X^2 \to \min, \quad \text{subject to} \quad D(F) \tag{2.30}$$

with the regularization parameter $\alpha > 0$ are exploited to find stable approximate solutions of an ill-posed operator equation (1.2). The smaller the regularization parameter α is chosen, the 'closer' the original and the auxiliary problem are related, but the more instable and highly oscillating the solution of the auxiliary problem will become. In general, α has to be selected such that an appropriate trade-off between stability and approximation is realized. In our compactification approach using upper bounds c of the total variation the inverse value $\frac{1}{c}$ plays a comparable role. In fact, if we consider small values $\frac{1}{c}$, then highly oscillating functions with large total variation values are admissible. On the other hand, for small values c the solutions obtained cannot oscillate very much, and the approximate solutions will be computed in a more stable way. However, if c is selected too small, then it may occur that the (unknown) exact solution is not an element of the set D. In such a case we would 'overregularize' the autoconvolution equation. By controlling the upper bound c of total variation we are able to suppress oscillations. Compared to the frequently used compactification in L^p by using monotonicity constraints and lower and upper bounds for the function values (see Section 4) the approach of this section allows us to handle a more comprehensive class of (also non-monotone) functions. A numerical case study presented in Section 5 will illustrate the theoretical results of this section and some specific effects of the discretized solution of the autoconvolution equation under total variation constraints.

In the case $p = \infty$ we cannot assert convergence under our assumption of bounded total variation. If the solution x^* has a jump point, then $\|x_\eta - x^*\|_{L^\infty(0,1)} \to 0$ as $\eta \to 0$ is not true in general.

3 The Sobolev Space Case

In [8] it was already mentioned that the operator F of autoconvolution according to (1.3) mapping from $X := L^2(0,1)$ into the space $Y := L^2(0,1)$ is *non-compact*, but it becomes a *compact* operator if we change the problem to the Sobolev space $X := H^1(0,1) \cong W_2^1(0,1)$ of functions x with a quadratically integrable generalized derivative x' and norm

$$\|x\|_{H^1(0,1)} = \left(\int_0^1 |x(t)|^2\, dt + \int_0^1 |x'(t)|^2\, dt \right)^{1/2}. \tag{3.1}$$

In both cases the autoconvolution equation is *locally ill-posed everywhere*. But for compact operators F, we have in general a stronger form of ill-posedness. If our pairs of spaces X and Y are Hilbert spaces, following the concept of [8] (see also [9, Sect. 2.2.2]) we can express the *local degree of ill-posedness* μ $(0 \le \mu \le \infty)$ of the autoconvolution equation in a solution point x^* by the *decay rate* of the *singular value sequence* $\sigma_1 \ge \sigma_2 \ge ... \ge \sigma_i \ge ... > 0$ tending to zero as $i \to \infty$ of the *Fréchet derivative* $F'(x^*)$ in the form

$$\mu := \sup\{\nu : \sigma_i = O(i^{-\nu}) \text{ as } i \to \infty\}, \tag{3.2}$$

where this linear operator given by $F'(x^*)h = 2h * x^*$ is *compact*. Since the compact embedding operator from $H^1(0,1)$ into $L^2(0,1)$ has a sequence of singular values $\kappa_1 \ge \kappa_2 \ge ... \ge \kappa_i \ge ... > 0$ tending to zero with a rate $\kappa_i \sim 1/i$ as $i \to \infty$, for the Sobolev space $X := H^1(0,1)$ under consideration in this section the ill-posedness degree grows at least by one (cf. Hofmann and Tautenhahn [10]) compared to the $L^2(0,1)$ case of Section 2. Thus, for a compactification in $H^1(0,1)$ 'stronger' restrictions on the admissible solutions x are necessary. However, our aim in this section is also stronger, namely to obtain convergence of approximate solutions x_η to x^* in the $H^1(0,1)$-norm (3.1).

Here we consider, for given constants a_1, a_2, b_1, b_2 and c with

$$0 < a_1 < b_1,\ a_2 < b_2, \tag{3.3}$$

the domain

$$D := \left\{ x : [0,1] \to [a_1, b_1],\ \begin{array}{cc} \exists x' : [0,1] \to [a_2, b_2], & x' \text{ left-continuous for } t \in (0,1], \\ T(x') \le c, & x' \text{ right-continuous for t=0} \end{array} \right\}, \tag{3.4}$$

where the function $x'(t)$ $(0 \le t \le 1)$ a.e. in $[0,1]$ coincides with a derivative of $x(t)$ in the classical sense. Obviously we get $D \subset H^1(0,1)$ and hence every function $x \in D$ with D from (3.3) – (3.4) is continuous. In analogy to Lemma 2.3 we have in the Sobolev space case:

Lemma 3.1 *The domain D from (3.3) – (3.4) is a compact subset of $H^1(0,1)$ with $D \subset D_0^+$.*

In contrast to the L^p-case the restriction of the total variation, here $T(x') \le c$, is only needed to show the compactness of the domain D. It has no relevance for the convergence of the images $F(x_\eta)$ of approximate solutions x_η to $F(x^*)$ in $L^2(0,1)$ as η tends to zero.

The discretization of the autoconvolution problem (1.1) – (1.3), where the operator F from (1.3) maps in the form

$$F : D \subset H^1(0,1) \to L^2(0,1) \tag{3.5}$$

and where the domain D is defined by (3.3) – (3.4) will be performed similar to the $L^p(0,1)$ case. However, piecewise constant functions are not in $H^1(0,1)$. Therefore, we use continuous piecewise linear approximate functions. Here, let (in contrast to Section 2)

$$t_j := jh \quad (j = 0, ..., n)$$

denote the $n+1$ nodes subdividing the interval $[0,1]$, and again $I_j = ((j-1)h, jh]$. Furthermore, the x_j again denote approximate values of $x(t_j)$. As the discrete autoconvolution operator we introduce here:

$$\underline{F} : \mathbb{R}^{n+1} \to \mathbb{R}^n, \tag{3.6}$$

where $\underline{F}(\underline{x}) = (z_1, ..., z_n)^T$ and for $i = 1, 2, ..., n$:

$$z_i = \int_0^{ih} (E_1(\underline{x}))\,(ih-t)\;(E_1(\underline{x}))\,(t)\,dt = \sum_{j=1}^{i} \frac{h}{6}\,(2x_{i-j}x_j + x_{i-j+1}x_j + x_{i-j}x_{j-1} + 2x_{i-j+1}x_{j-1})\,. \tag{3.7}$$

By $E_1 : \mathbb{R}^{n+1} \to H^1(0,1)$ we denote in contrast to Section 2 the operator of piecewise linear interpolation according to

$$(E_1(\underline{x}))(t) := \frac{t-jh}{h}(x_j - x_{j-1}) + x_j \quad (t \in I_j,\ j = 1, ..., n). \tag{3.8}$$

For noisy data (see (2.15)) we search for a minimizer

$$\underline{x}^{opt} = (x_0^{opt}, x_1^{opt}, ..., x_n^{opt})^T$$

of the least-squares problem (2.20) with M from

$$M := \left\{ \underline{x} \in \mathbb{R}^{n+1} : \begin{array}{c} 0 < a_1 \le x_i \le b_1 \ (i = 0, ..., n), \\ ha_2 \le x_i - x_{i-1} \le hb_2 \ (i = 1, ..., n), \end{array} \ \sum_{i=1}^{n-1} |x_{i+1} - 2x_i + x_{i-1}| \le hc \right\}. \tag{3.9}$$

With the same arguments as before it follows that (2.20) is solvable. The choice of $\underline{F}$ is due to the fact that we have to guarantee the validity of formula (2.24) with F_η from (2.23).

By setting for the approximate solution

$$x_\eta := E_1(\underline{x}^{opt}), \tag{3.10}$$

where $\eta = (\delta, h)^T$, we also have $x_\eta \in D$ with D according to (3.3)-(3.4). Moreover, it can be shown that as in Lemma 2.7 we have $\|F(x_\eta) - F(x^*)\|_{L^2(0,1)} \to 0$ for $\eta \to 0$. The proof dealing with the $H^1(0,1)$ approximation of functions by linear splines is omitted here. Using again Lemma 2.4 with $X := H^1(0,1)$ and $Y := L^2(0,1)$ we obtain:

Theorem 3.2 *Consider the autoconvolution problem (1.1) – (1.3) with $D(F) := D$ from (3.3) – (3.4) and denote by $x^* \in D$, for given right-hand side $y^* \in F(D(F))$, the unique solution of the autoconvolution equation. Then the family of approximate solutions x_η converges to the solution x^* of (1.2):*

$$\|x_\eta - x^*\|_{H^1(0,1)} \to 0 \quad as \quad \eta \to 0. \tag{3.11}$$

4 Monotonicity Constraints

In this section we deal with solutions of the autoconvolution equation subject to the set of monotone and uniformly bounded functions considered as a particular subset of the functions possessing a bounded total variation.

First we consider the domain

$$D := \{x : 0 \le x(t) \le b,\ t \in [0,1],\ x \text{ non-increasing}\} \tag{4.1}$$

forming a compact subset in $L^p(0,1)$, $1 \le p < \infty$. Then the operator F from (2.10) is also injective, since $D \subset D_0^+ \cup \{0\}$ and $x(t) = 0$ $(0 \le t \le 1)$ is the only function of D according to (4.1) with $x(0) = 0$. The discretization of this monotonicity case is completely the same as given in Section 2 for the total variation case with the exception of the fact that we have to introduce

$$M := \{\underline{x} \in \mathbb{R}^n : 0 \le x_n \le ... \le x_1 \le b\}. \tag{4.2}$$

replacing (2.21). Since each monotone function is of bounded variation, we obtain the convergence results of Section 2 with $c = b$ and $a = 0$.

Now we change to the case of non-decreasing solutions, where

$$D := \{x : 0 \le x(t) \le b,\ t \in [0,1],\ x \text{ non-decreasing}\} \tag{4.3}$$

and

$$M := \{\underline{x} \in \mathbb{R}^n : 0 \le x_1 \le ... \le x_n \le b\}. \tag{4.4}$$

The set D from (4.3) is also compact in $L^p(0,1)$, but the injectivity of F fails (cf. [8]). Because of that we have to distinguish two cases:

On the one hand let $y \in R_0^+$, i.e. $y(s) > 0$ if $s > 0$. Then the corresponding solution $x^*(t)$ is uniquely determined from y a.e. in $[0,1]$ and $\|F(x_\eta) - F(x^*)\|_{L^2(0,1)} \to 0$ for $\eta \to 0$ also implies $\|x_\eta - x^*\|_{L^p(0,1)} \to 0$, since Tikhonov's lemma (see Lemma 2.4) in fact only needs the local injectivity condition $F(x) = F(x^*)$ $(x \in D) \Longrightarrow x = x^*$.

On the other hand, let $y \in R_\varepsilon^+$ for $\varepsilon > 0$, i.e. $y(s) = 0$ if $s \in [0,\varepsilon]$. As shown in [8], in such a case the autoconvolution operator F is *non-injective* and it holds:

$$x^*(t) = \begin{cases} 0 & \text{a.e. in } [0, \frac{\varepsilon}{2}] \\ \text{uniquely determined} & \text{a.e. in } [\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}] \\ \text{arbitrarily non-negative} & \text{in } [1-\frac{\varepsilon}{2}, 1] \end{cases}.$$

Consequently, we have $x^* \in D_{\frac{\varepsilon}{2}}^+$. Since the values $x^*(t)$ do not depend on y for $t \in [1-\frac{\varepsilon}{2}, 1]$, we cannot expect any information about the solution in this subinterval from the data. Therefore, it makes sense to solve the equation (1.1) only on the interval $[\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}]$. We will show that this case is reducible to the already treated case $y \in R_0^+$. Because of this we define the operator $F_\varepsilon : L^p(\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}) \to L^2(\varepsilon, 1)$ as

$$[F_\varepsilon(x)](s) := \int_{\frac{\varepsilon}{2}}^{s-\frac{\varepsilon}{2}} x(s-t)x(t)dt.$$

Then we have $[F(x)](s) = [F_\varepsilon(x)](s)$ for $\frac{\varepsilon}{2} \le s \le 1-\frac{\varepsilon}{2}$. By using the transformations

$$\tilde{t} := \frac{t-\frac{\varepsilon}{2}}{1-\varepsilon}, \quad \tilde{s} := \frac{s-\varepsilon}{1-\varepsilon},$$

and

$$\tilde{x}(\tilde{t}) := x((1-\varepsilon)\tilde{t} + \frac{\varepsilon}{2}) = x(t), \quad \tilde{y}(\tilde{s}) := y((1-\varepsilon)\tilde{s}+\varepsilon) = y(s), \quad \tilde{F}_\varepsilon(\tilde{x}) := F_\varepsilon(x),$$

we obtain an operator $\tilde{F}_\varepsilon : L^p(0,1) \to L^2(0,1)$ defined by

$$[\tilde{F}_\varepsilon(\tilde{x})](\tilde{s}) := (1-\varepsilon)\int_0^{\tilde{s}} \tilde{x}(\tilde{s}-\tilde{t})\tilde{x}(\tilde{t})d\tilde{t}.$$

Then we get $\tilde{x} \in L^p(0,1)$ if $x \in L^p(0,1)$, and instead of (1.2) we have to solve the equation $\tilde{F}_\varepsilon(\tilde{x}) = \tilde{y}$ now. From $y \in R_\varepsilon^+$ and $x \in R_{\frac{\varepsilon}{2}}^+$ it follows that $\tilde{y} \in R_0^+$ and $\tilde{x} \in D_0^+$, respectively. Hence we have $\tilde{F}_\varepsilon(\tilde{x}) = (1-\varepsilon)F(\tilde{x})$ for all $\tilde{x} \in D_0^+$. Therefore, we can proceed as in the injective case and compute converging approximate solutions $\tilde{x}_\eta$. Then we transform back to the interval $[\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}]$ and obtain approximate solutions with satisfactory properties on this interval, where the performed linear transformation retains the monotonicity. Finally we extend the solution by zero on the interval $[0, \frac{\varepsilon}{2})$. On the other remaining subinterval $(1-\frac{\varepsilon}{2}, 1]$ the solution can be extended arbitrarily provided that the monotonicity requirement is satisfied. Unfortunately, the value of ε is unknown if only discrete noisy data are given. In some situations, however, this value can be estimated and the transformation procedure becomes applicable.

5 Numerical Examples

In the concluding section we present some case studies on the behaviour of approximate discrete least-squares solutions to the autoconvolution equation (1.1) from noisy data, where we follow the approach of Section 2.

The first study is devoted to the case of a *continuous*, but *non-monotone* exact solution. We use the example

$$x^*(t) = -3t^2 + 3t + \frac{1}{4} = 1 - 3\left(t - \frac{1}{2}\right)^2 \qquad (0 \le t \le 1) \tag{5.1}$$

with the right-hand side

$$y^*(s) = \frac{3}{10}s^5 - \frac{3}{2}s^4 + s^3 + \frac{3}{4}s^2 + \frac{1}{16}s \quad (0 \le s \le 1) \tag{5.2}$$

and obtain $a := 0.25 \le x^*(t) \le b := 1$, $T(x^*) = 1.5$ and $x^* \in D$ with D from (2.5) - (2.6). The noisy data $\underline{\hat{y}}$ were generated by adding normally distributed pseudorandom numbers with zero mean and standard deviation σy_i (σ fixed) to the discrete values y_i of (5.2). We used varying values c as upper bounds for the total variation of the discretized solutions.

The nonlinear optimization problem (2.20) was numerically solved by a Gauss-Newton code. In the case of unacceptable Gauss-Newton steps this code uses the Marquardt method. The theory of this procedure is due to [3, pp. 348-368] (for the algorithm see [3, pp. 369-383]). We used penalty terms to handle the constraints of D. In all figures presented below the solid lines give the exact solution x^* according to (5.1), whereas the lines with small circles express the approximate solutions x_η such that every circle corresponds to a grid point of discretization.

In the Figures 1 and 2 we compare approximate solutions x_η in the case of unperturbed data ($\sigma = 0$) using $n = 50$ grid points and different bounds c for the total variation. For an appropriate choice $c = 1.5$ associated with the really arising total variation level, the approximate solution is very good in the noiseless case (see Figure 1), whereas an underestimated value $c = 0.8 < T(x^*)$ corresponds to an overregularized solution (see Figure 2), which is much too 'flat' compared to the function x^* to be determined.

Now we turn to the case of noisy data. For all computations in the context of the Figures 3 – 6 a per mille noise level $\sigma = 10^{-3}$ was used. We begin with a situation (see Figure 3), where the total variation bound was omitted ($c = \infty$). Then the set M of admissible discrete solutions contains strongly oscillating vectors. Especially for t from the right half-interval of $[0, 1]$ the quality of the approximate solutions may be very bad in that situation.

The Figure 4 illustrates in a rather convincing manner the utility of the total variation approach presented above in Section 2 for handling noisy data. In particular, the approximation quality of x_η in Figure 4 with $c = 1.5$ at the *right end* of the interval is much better than in Figure 3. We can motivate this right-end effect as follows: By the autoconvolution of a function $x(t)$ $(0 \le t \le 1)$ the values $x(t)$ for small t influence the function values $y(s)$ in some sense more than the values $x(t)$ with t close to 1. Namely, $x(t)$ only influences $y(s)$ for $s > t$. As a consequence, the reconstruction of $x(t)$ from y is more stable for smaller t, since then the function $y(s) = [x * x](s)$ $(0 \le s \le 1)$ has collected more information about the value $x(t)$ to be determined. In the case of overregularization (c is selected too small compared to $T(x^*)$), this phenomenon may cause large reconstruction errors specifically at the right end of the interval $[0, 1]$ (see Figure 5 with $c = 1.0$).

We should mention that the analysis of the problem based on Lemma 2.4 does not provide any rate of convergence for the solution error $\|x_\eta - x^*\|_{L^p(0,1)}$ depending on $\|F(x_\eta) - F(x^*)\|_{L^2(0,1)}$. On the other hand, Lemma 2.7 shows that the order of magnitude for the discrepancy norm $\|F(x_\eta) - F(x^*)\|_{L^2(0,1)}$ corresponds with the maximum $\max(h\bar{b}^2, h\bar{b}\bar{c}, \delta)$, where $h := 1/n$, $\bar{b} := \sup\limits_{\substack{x \in D \\ t \in [0,1]}} x(t)$ and $\bar{c} := \sup\limits_{\substack{x \in D \\ t \in [0,1]}} T(x)$. For sufficiently large n this discrepancy norm is dominated by the noise level δ, or in our case study by the value σ. So we can see comparing the Figures 4 and 6 that different discretization levels $n = 50$ and $n = 25$ yield approximate solutions with nearly the same accuracy provided that the noise level ($\sigma = 10^{-3}$) does not change.

To a second study we have been motivated by numerical experiments carried out by Kutsche in her thesis [11]. There it was shown that the constraint of bounded variation is very useful in the L^1-approximation of piecewise continuous solutions to Abel integral equations. We now will demonstrate the effects of using the least-squares method under total variation constraints in the case of *non-smooth* functions possessing jumps. Therefore we consider as the exact solution the step function

$$x^*(t) = \begin{cases} 0.5 & \text{if } 0 \le t \le 0.5 \\ 0.25 & \text{if } 0.5 < t \le 0.8 \\ 0.75 & \text{if } 0.8 < t \le 1 \end{cases} \tag{5.3}$$

with the right-hand side

$$y^*(s) = \begin{cases} 0.25t & \text{if } 0 \le t \le 0.5 \\ 0.125 & \text{if } 0.5 < t \le 0.8 \\ 0.5t - 0.275 & \text{if } 0.8 < t \le 1 \end{cases} . \tag{5.4}$$

The exact solution x^* is discontinuous, non-monotone but a function of bounded variation. Its total variation can easily computed as $T(x^*) = 0.75$. The function x^* is bounded, positive and left-continuous on the whole interval $[0, 1]$. Therefore the requirements of the set D from (2.5) - (2.6) are fulfilled.

We will now compare the approximate solutions of this example for different choices of the parameter c. Let the number of discretization points $n = 50$ and the value $\sigma = 10^{-2}$ of noise be constant throughout this study. Then we are able to control the solution by changing the parameter c.

In the Figures $7 - 10$ the graphs of both the numerical solution x_η and the exact solution x^* (bold line) are drawn as piecewise constant functions. In our first example (Figure 7) we computed the solution without any total variation restriction ($c = \infty$). The solution is – as in the first example – rather bad and highly oscillating. However, it is to mention that the jumps of x^* are reconstructed relatively good in this case. In Figure 8 the situation $c = T(x^*) = 0.75$ is illustrated. Here the solution is much smoother than in the unconstrained case, but the points with maximal approximation errors are now the jumps at $t = 0.5$ and $t = 0.8$. In these points the approximate solution is 'oversmoothed'. This depends on the fact that the smoothing effect of regularization acts uniformly on the whole interval $[0, 1]$, but the character of a jump function does not correspond with this property. Therefore the jumps are blurred by that choice of c. Moreover, the 'right-end effect' discussed above is superposed and leads to growing errors near $t = 1$.

Finally, in Figure 9 with $c = 0.5$ and Figure 10 with $c = 2$ we demostrate two more situations. If c underestimates the value $T(x^*)$, then the effect of blurring the jumps is still more pronounced. On the other hand, the admissible oscillation level grows if c overestimates $T(x^*)$. In that case, however, the location of jumps can be determined rather precise. That means, if one supposes that the exact solution is a step function, then it is recommended to choose c not too small. This allows some oscillations around the exact solution whose amplitudes are small if c is not chosen much too large.

References

[1] Acar, R.; Vogel, C.R.: Analysis of total variation penalty methods. *Inverse Problems* **10** (1994), 1217-1229.

[2] Baumeister, J.: *Stable Solution of Inverse Problems*. Vieweg, Braunschweig 1987.

[3] Dennis, J.E.; Gay, D.M.; Welsch, R.E.: An adaptive nonlinear least–squares algorithm. *ACM Trans. Math. Softw.* **7** (1981), 348-368.

[4] Engl, H.W.; Hofmann, B.; Zeisel, H.: A decreasing rearrangement approach for a class of ill-posed nonlinear integral equations. *Journal of Integral Equations and Applications* **5** (1993), 443-463.

[5] Fleischer, G; Hofmann, B.: On inversion rates for the autoconvolution equation. *Inverse Problems* **12** (1996), 419-435.

[6] Gellrich, C.; Hofmann, B.: A study of regularization by monotonicity. *Computing* **50** (1993), 105-125.

[7] Gorenflo, R.: Computation of rough solutions of Abel integral equations. In: Inverse and Ill-Posed Problems (Eds.: H.W. Engl, C.W. Groetsch), Academic Press, Boston, 1987, 195-210.

[8] Gorenflo, R.; Hofmann, B.: On autoconvolution and regularization. *Inverse Problems* **10** (1994), 353-373.

[9] Hofmann, B.: *Regularization for Applied Inverse and Ill-Posed Problems*. B.G. Teubner, Leipzig 1986.

[10] Hofmann, B.; Tautenhahn, U.: On ill-posedness measures and space change in Sobolev scales. Paper submitted.

[11] KUTSCHE, C.: Produktquadraturverfahren für nicht-glatte Lösungen Abelscher Integralgleichungen erster und zweiter Art. Dissertation, Freie Universität Berlin, FB Mathematik und Informatik, Berlin 1994.

[12] NATANSON, I.P.: *Theorie der Funktionen einer reellen Veränderlichen*. Akademie-Verlag, Berlin, 1969.

[13] TIKHONOV, A.N; GONCHARSKY, A.V.; STEPANOV, V.V.; YAGOLA, A.G.: *Regularizing algorithms and a priori information* (in Russian). Nauka, Moscow, 1983.

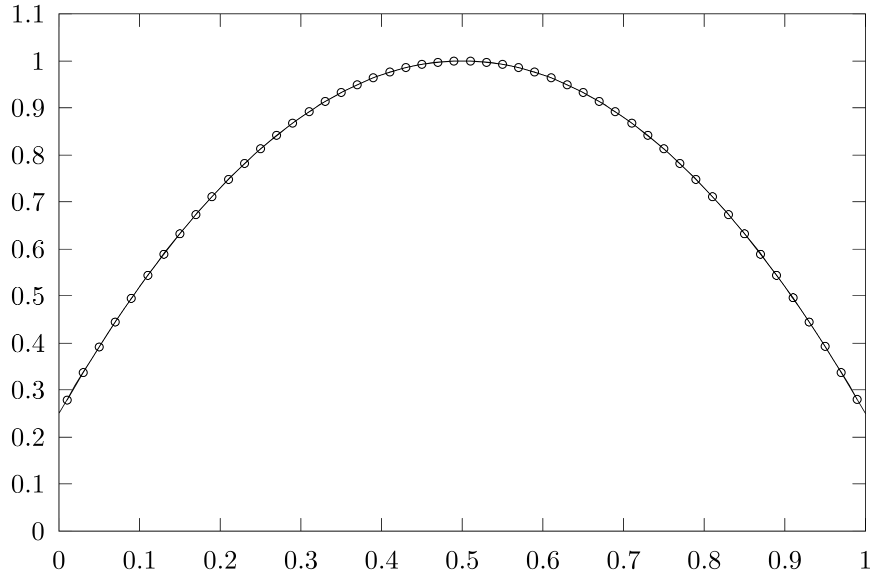

Figure 1: Least-squares solution of $\underline{F}(\underline{x}) = \underline{\hat{y}}$, $\sigma = 0$, $n = 50$, $c = 1.5$

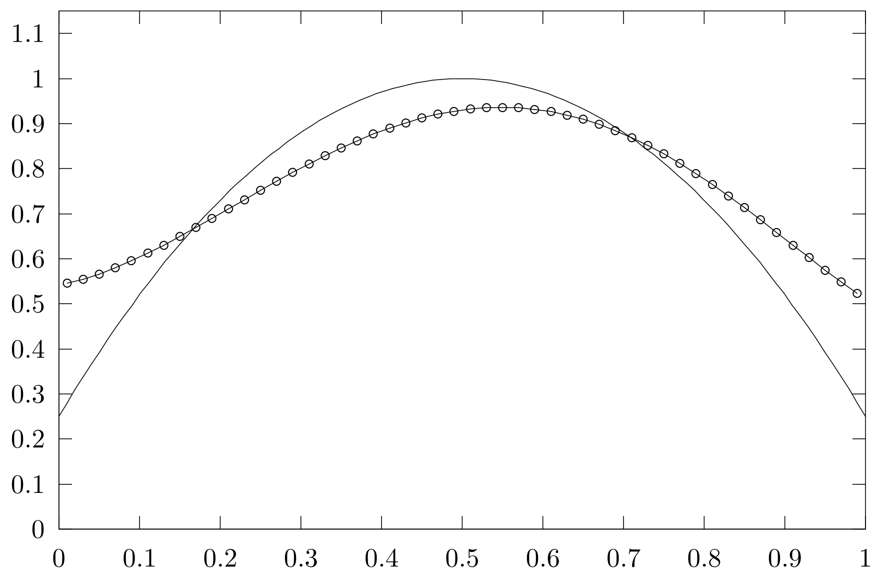

Figure 2: Least-squares solution of $\underline{F}(\underline{x}) = \underline{\hat{y}}$, $\sigma = 0$, $n = 50$, $c = 0.8$

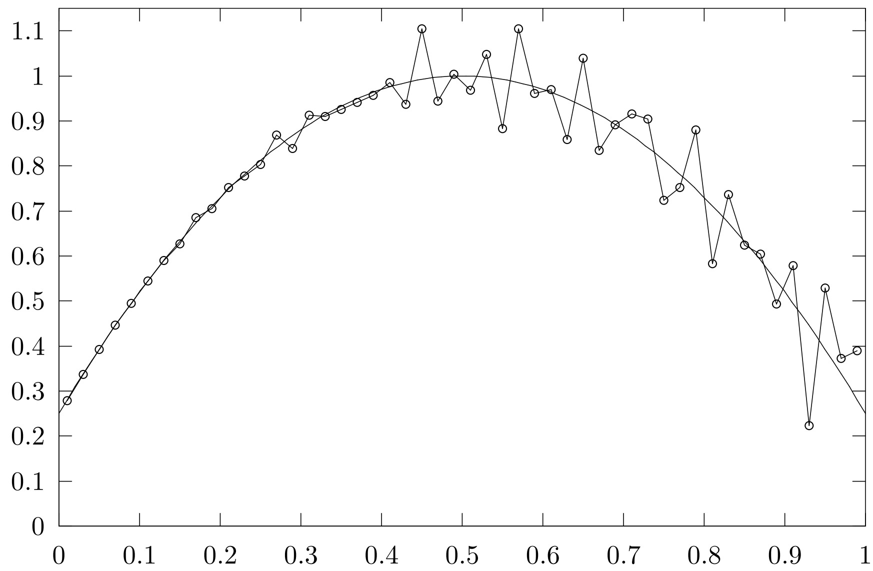

Figure 3: Least-squares solution of $\underline{F}(\underline{x}) = \underline{\hat{y}}$, $\sigma = 10^{-3}$, $n = 50$, $c = \infty$

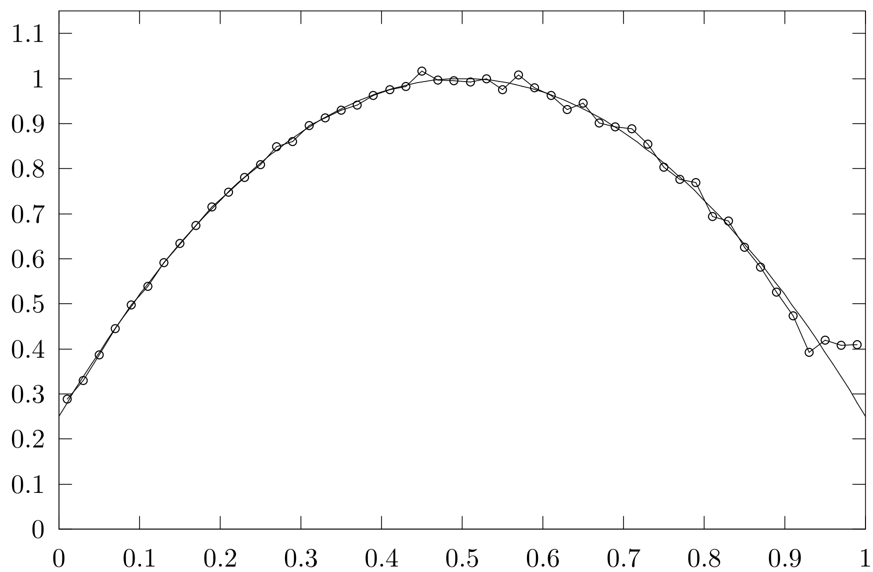

Figure 4: Least-squares solution of $\underline{F}(\underline{x}) = \underline{\hat{y}}$, $\sigma = 10^{-3}$, $n = 50$, $c = 1.5$

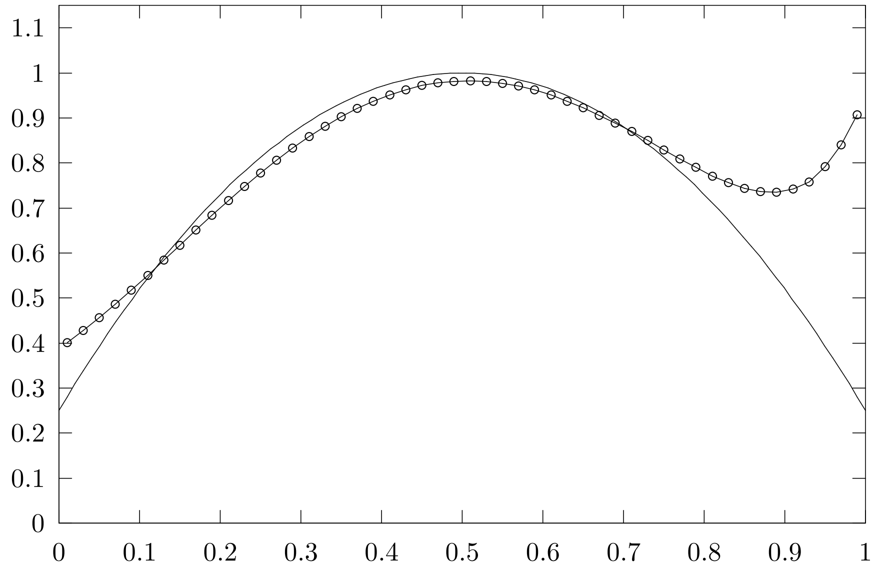

Figure 5: Least-squares solution of $\underline{F}(\underline{x}) = \hat{\underline{y}}$, $\sigma = 10^{-3}$, $n = 50$, $c = 1.0$, (inappropriate initial values used in Gauss-Newton-method)

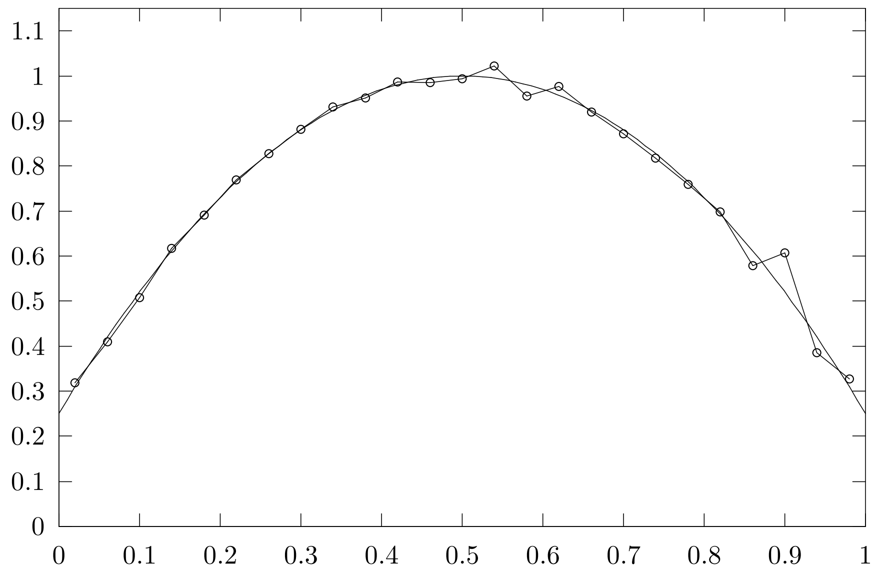

Figure 6: Least-squares solution of $\underline{F}(\underline{x}) = \hat{\underline{y}}$, $\sigma = 10^{-3}$, $n = 25$, $c = 1.5$

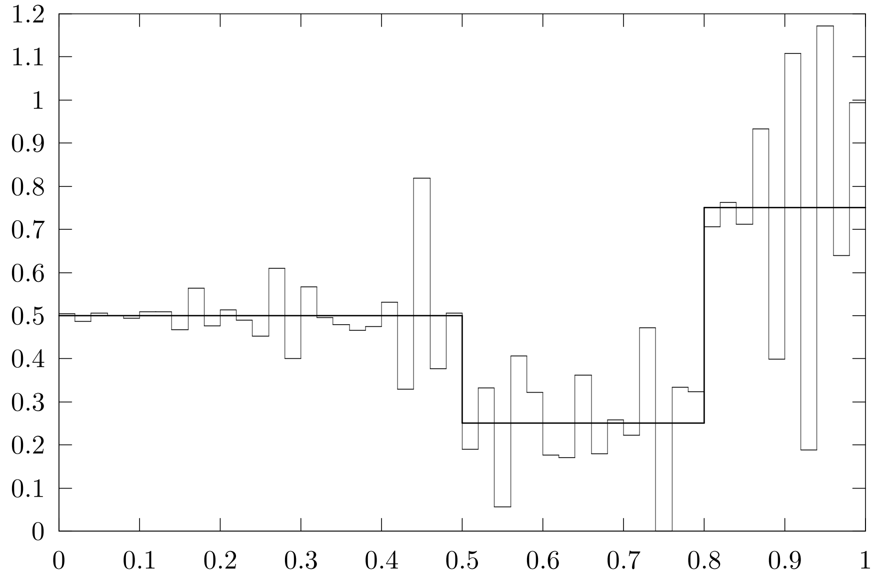

Figure 7: Least-squares solution of $\underline{F}(\underline{x}) = \underline{\hat{y}}$, $\sigma = 10^{-2}$, $n = 50$, $c = \infty$

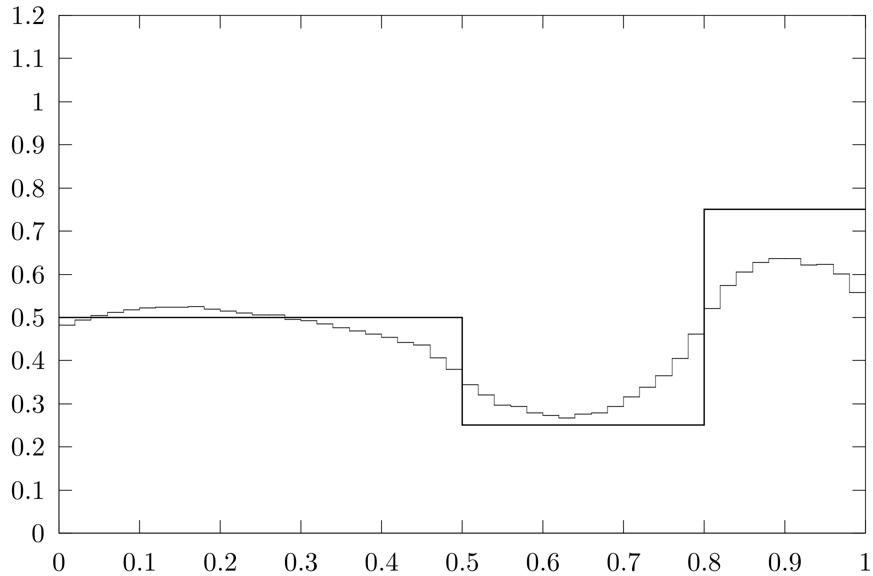

Figure 8: Least-squares solution of $\underline{F}(\underline{x}) = \underline{\hat{y}}$, $\sigma = 10^{-2}$, $n = 50$, $c = 0.75$

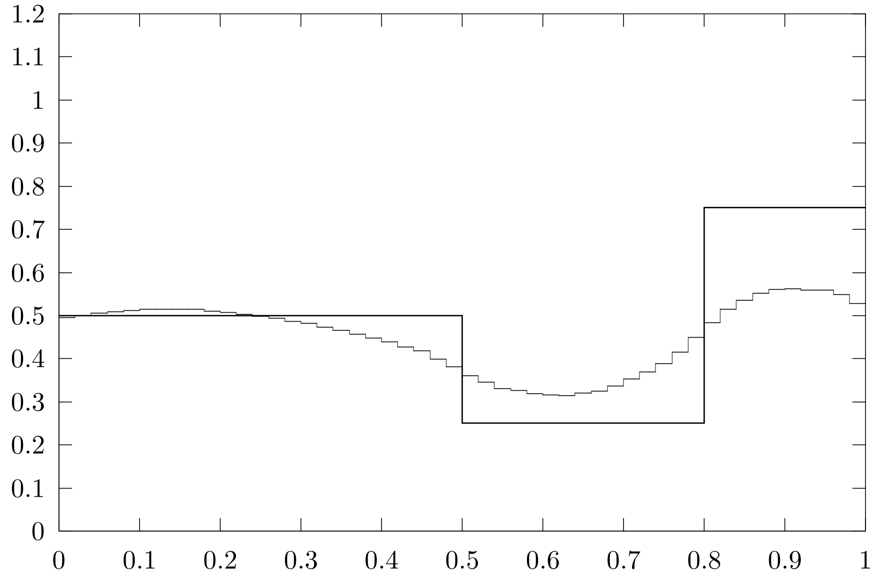

Figure 9: Least-squares solution of $\underline{F}(\underline{x}) = \underline{\hat{y}}$, $\sigma = 10^{-2}$, $n = 50$, $c = 0.5$

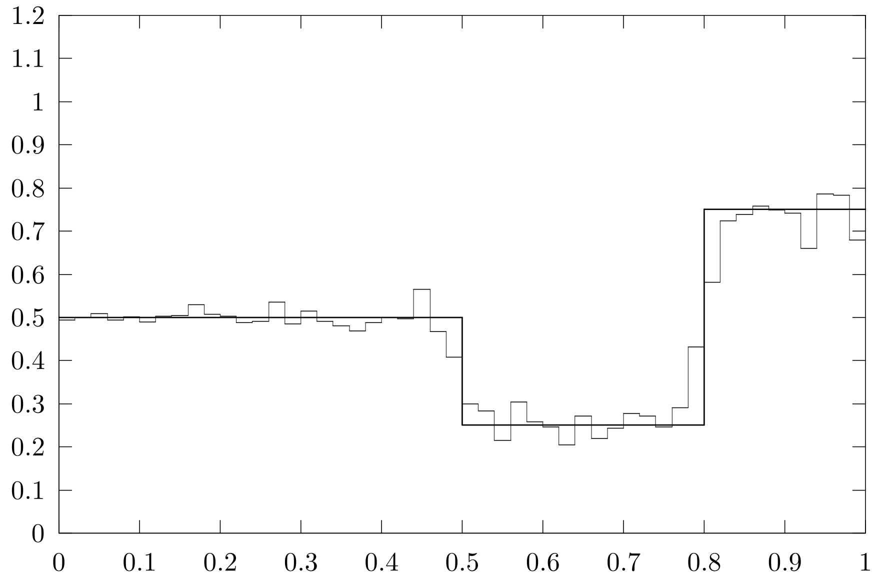

Figure 10: Least-squares solution of $\underline{F}(\underline{x}) = \underline{\hat{y}}$, $\sigma = 10^{-2}$, $n = 50$, $c = 2.0$